

Deriving conservation laws for ABS lattice equations from Lax pairs

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Abstract

In the paper we derive infinitely many conservation laws for the ABS lattice equations from their Lax pairs. These conservation laws can algebraically be expressed by means of some known polynomials. We also show that H1, H2, H3, Q1, Q2, Q3 and A1 equation in ABS list share a generic discrete Riccati equation.

Keywords: conservation laws, Lax pairs, ABS lattice equations, discrete integrable systems

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1 Introduction

The celebrated ABS list [1] consists of totally nine quadrilateral lattice equations which are consistent around a cube(CAC) (3D-consistency with extra restriction: quasi-linearity, D_4 symmetry and Tetrahedron property). A CAC equation itself provides a Bäcklund transformation(BT) as well as a Lax pair [1]. However, such a Lax pair seems hard to play a same role as useful as in continuous case, where from Lax pairs usually one can derive solutions either through Inverse Scattering Transform [2] or Darboux transformation [3], construct BTs [5], evolution equation hierarchy, commutative flows and infinitely many symmetries [4] and infinitely many conservation laws [5].

With regard to infinitely many conservation laws which serve as an integrability characteristic, there are many ways to derive them for continuous and semi-discrete integrable systems [5–10]. Using Lax pair is a simple way [5, 9, 10]. For ABS lattice equations, their conservation laws have been derived through a direct approach [11] based on the idea of [12], symmetry approach [13–15], Gardner’s approach (using BTs and initial conservation laws) [15, 16], and using quasi-difference operators and recursion operators [17], etc. In this paper, we will start from Lax pairs to derive infinitely many conservation laws for ABS lattice equations. In Ref. [9] (also see [10]) we introduce two kinds of techniques to construct conservation laws respectively for the Toda lattice and Ablowitz-Ladik system. In fact, Gardner method used in [16] is closely related to the technique used for the Ablowitz-Ladik system [9]. However, one will see that we can easily write out the so-called initial conservation laws from Lax pairs, and infinitely many

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conservations laws can algebraically be expressed by means of known polynomials. Besides, we also find many ABS lattice equations share a generic discrete Riccati equation.

We organize the paper as follows. In Sec.2 we introduce ABS list and the main idea of our approach. In Sec.3 first H1 equation serves as a detailed example to derive conservation laws. Then we list main results for H2, H3, Q1, Q2, Q3 and A1 equation. Finally, also in this section, we derive conservation laws for A2 and Q4 equation in a slightly different way but still starting from Lax pairs.

2 Preliminary and main idea

Let us start from the following quadrilateral equation

$$Q(u, \tilde{u}, \hat{u}, \hat{\tilde{u}}, p, q) = 0, \quad (2.1)$$

where

$$u = u(n, m), \quad \tilde{u} = E_n u = u(n+1, m), \quad \hat{u} = E_m u = u(n, m+1), \quad \hat{\tilde{u}} = u(n+1, m+1),$$

E_n and E_m respectively serve as shift operators in direction n and m , p and q are spacing parameters of direction n and m , respectively.

The ABS list reads [1]

$$(u - \hat{\tilde{u}})(\tilde{u} - \hat{u}) + q - p = 0, \quad (\text{H1})$$

$$(u - \hat{\tilde{u}})(\tilde{u} - \hat{u}) + (q - p)(u + \tilde{u} + \hat{u} + \hat{\tilde{u}}) + q^2 - p^2 = 0, \quad (\text{H2})$$

$$p(u\tilde{u} + \hat{u}\hat{\tilde{u}}) - q(u\hat{u} + \tilde{u}\hat{\tilde{u}}) + \delta(p^2 - q^2) = 0, \quad (\text{H3})$$

$$p(u + \hat{u})(\tilde{u} + \hat{\tilde{u}}) - q(u + \tilde{u})(\hat{u} + \hat{\tilde{u}}) - \delta^2 pq(p - q) = 0, \quad (\text{A1})$$

$$(q^2 - p^2)(u\tilde{u}\hat{u}\hat{\tilde{u}} + 1) + q(p^2 - 1)(u\hat{u} + \tilde{u}\hat{\tilde{u}}) - p(q^2 - 1)(u\tilde{u} + \hat{u}\hat{\tilde{u}}) = 0, \quad (\text{A2})$$

$$p(u - \hat{u})(\tilde{u} - \hat{\tilde{u}}) - q(u - \tilde{u})(\hat{u} - \hat{\tilde{u}}) + \delta^2 pq(p - q) = 0, \quad (\text{Q1})$$

$$p(u - \hat{u})(\tilde{u} - \hat{\tilde{u}}) - q(u - \tilde{u})(\hat{u} - \hat{\tilde{u}}) + pq(p - q)(u + \tilde{u} + \hat{u} + \hat{\tilde{u}}) - pq(p - q)(p^2 - pq + q^2) = 0, \quad (\text{Q2})$$

$$(q^2 - p^2)(u\hat{\tilde{u}} + \tilde{u}\hat{u}) + q(p^2 - 1)(u\tilde{u} + \hat{u}\hat{\tilde{u}}) - p(q^2 - 1)(u\hat{u} + \tilde{u}\hat{\tilde{u}}) - \delta^2(p^2 - q^2)(p^2 - 1)(q^2 - 1)/(4pq) = 0, \quad (\text{Q3})$$

$$p(u\tilde{u} + \hat{u}\hat{\tilde{u}}) - q(u\hat{u} + \tilde{u}\hat{\tilde{u}}) = \frac{pQ - qP}{1 - p^2q^2} [(\hat{u}\hat{\tilde{u}} + u\tilde{u}) - pq(1 + u\tilde{u}\hat{u}\hat{\tilde{u}})], \quad (\text{Q4})$$

where Q4 equation is of the form given by Hietarinta [18], and in the equation $P^2 = p^4 - kp^2 + 1$, $Q^2 = q^4 - kq^2 + 1$ with parameter δ .

If (2.1) is a CAC equation, then it is easy to write out its BT

$$Q(u, \tilde{u}, \bar{u}, \tilde{\bar{u}}, p, r) = 0, \quad (2.2a)$$

$$Q(u, \bar{u}, \hat{u}, \hat{\bar{u}}, r, q) = 0, \quad (2.2b)$$

where r serves as a soliton parameter, and if u solves (2.1), so does \bar{u} . Replacing \bar{u} by ϕ_1/ϕ_2 , the above BT can be rewritten in terms of $\phi = (\phi_1, \phi_2)^T$ as the following,

$$\tilde{\phi} = \frac{\beta}{\sqrt{|M|}} M(u, \tilde{u}, p, r) \phi, \quad (2.3a)$$

$$\hat{\phi} = \frac{\gamma}{\sqrt{|N|}} N(u, \hat{u}, q, r) \phi, \quad (2.3b)$$

where M and N are 2×2 matrices, $\frac{1}{\sqrt{|M|}}$ and $\frac{1}{\sqrt{|N|}}$ are to guarantee the consistency, and β and γ are constants that can be arbitrary but play key roles to get a solvable discrete Riccati equation. Equations (2.3) can be a Lax pair of equation (2.1) and r serves as a spectral parameter. A conservation law of (2.1) is defined by

$$\Delta_m F(u) = \Delta_n J(u), \quad (2.4)$$

where $\Delta_n = E_n - 1$, $\Delta_m = E_m - 1$ and u solves the equation (2.1).

From the Lax pair (2.3) we construct a formal conservation law in the following way. First we define

$$\theta = \frac{\tilde{\phi}_2}{\phi_2}, \quad \eta = \frac{\hat{\phi}_2}{\phi_2}. \quad (2.5)$$

Then noting that

$$\ln \theta = \Delta_n \ln \phi_2, \quad \ln \eta = \Delta_m \ln \phi_2, \quad (2.6)$$

we immediately reach to

$$\Delta_m \ln \theta = \Delta_n \ln \eta, \quad (2.7)$$

which is a formal conservation law of the lattice equation related to the Lax pair (2.3).

We find that for lattice equations H1, H2, H3, A1, Q1, Q2, and Q3 in ABS list, θ satisfies a discrete Riccati equation of the following form,*

$$\tilde{\mu} \tilde{\theta} \theta = (u - \tilde{u}) \theta - \varepsilon^2 \mu, \quad (2.9)$$

where μ is a function of u, \tilde{u} related to considered equations, ε is a constant related to p, r . It is not difficult to verify that

Proposition 1. *The discrete Riccati equation (2.9) is solved by*

$$\theta = \varepsilon^2 \rho \left(1 + \sum_{j=1}^{\infty} \theta_j \varepsilon^{2j} \right), \quad (2.10a)$$

with

$$\rho = \frac{\mu}{u - \tilde{u}}, \quad (2.10b)$$

$$\theta_{j+1} = \frac{\tilde{\mu} \tilde{\rho}}{u - \tilde{u}} \sum_{i=0}^j \tilde{\theta}_i \theta_{j-i}, \quad j = 0, 1, 2, \dots, \quad (\theta_0 = 1). \quad (2.10c)$$

* In fact, supposing that $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in (2.3a), one can always get

$$\sqrt{|M|} \tilde{\phi}_2 = \beta \tilde{C} \left(\frac{A}{C} - \frac{\tilde{D}}{\tilde{C}} \right) \tilde{\phi}_2 - \beta^2 \sqrt{|M|} \frac{\tilde{C}}{C} \phi_2, \quad (2.8)$$

which, divided by ϕ_2 , yields a discrete Riccati equation of θ .

This gives an explicit form of θ , but it is not enough to get infinitely many conservation laws from (2.7). We still need an explicit η . However, we can not insert η into a Riccati equation similar to (2.9) with same ε because ε is independent of q . Fortunately, from the Lax pair (2.3) we may find the following relation

$$\eta = \omega(\sigma\theta + 1), \quad (2.11)$$

where both ω and σ are functions of $(u, \tilde{u}, \hat{u}, p, q)$ and they satisfy

$$\frac{1}{\omega(u, \tilde{u}, \hat{u}, p, q)} = -\sigma(u, \hat{u}, \tilde{u}, q, p). \quad (2.12)$$

Thus, substituting the above η together with (2.10) into the formal conservation law (2.7) it is possible to get explicit infinitely many conservation laws. To do that, we make use of the following expansion formula.

Proposition 2. *The following expansion holds,*

$$\ln\left(1 + \sum_{i=1}^{\infty} t_i k^i\right) = \sum_{j=1}^{\infty} h_j(\mathbf{t}) k^j, \quad (2.13a)$$

where

$$h_j(\mathbf{t}) = \sum_{\|\boldsymbol{\alpha}\|=j} (-1)^{|\boldsymbol{\alpha}|-1} (|\boldsymbol{\alpha}| - 1)! \frac{\mathbf{t}^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!}, \quad (2.13b)$$

and

$$\mathbf{t} = (t_1, t_2, \dots), \quad \boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots), \quad \alpha_i \in \{0, 1, 2, \dots\}, \quad (2.13c)$$

$$\mathbf{t}^{\boldsymbol{\alpha}} = \prod_{i=1}^{\infty} t_i^{\alpha_i}, \quad \boldsymbol{\alpha}! = \prod_{i=1}^{\infty} (\alpha_i!), \quad |\boldsymbol{\alpha}| = \sum_{i=1}^{\infty} \alpha_i, \quad \|\boldsymbol{\alpha}\| = \sum_{i=1}^{\infty} i\alpha_i. \quad (2.13d)$$

The first few of $\{h_j(\mathbf{t})\}$ are

$$h_1(\mathbf{t}) = t_1, \quad (2.14a)$$

$$h_2(\mathbf{t}) = -\frac{1}{2}t_1^2 + t_2, \quad (2.14b)$$

$$h_3(\mathbf{t}) = \frac{1}{3}t_1^3 - t_1t_2 + t_3, \quad (2.14c)$$

$$h_4(\mathbf{t}) = -\frac{1}{4}t_1^4 + t_1^2t_2 - t_1t_3 - \frac{1}{2}t_2^2 + t_4. \quad (2.14d)$$

We note that $\{h_j(\mathbf{t})\}$ also satisfy

$$\partial_{t_s} h_{i+s}(\mathbf{t}) = \partial_{t_j} h_{i+j}(\mathbf{t}), \quad \text{for } i, j, s \in \mathbb{Z}^+, \quad (2.15)$$

and $\{h_j(\mathbf{t})\}$ are different from the Schur function (See [19]).

Proof. Let us first prove the following expansion:

$$\left(\sum_{i=1}^{\infty} y_i\right)^s = s! \sum_{|\boldsymbol{\alpha}|=s} \frac{\mathbf{y}^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!}, \quad (s \in \mathbb{Z}^+), \quad (2.16a)$$

$$\mathbf{y} = (y_1, y_2, \dots), \quad \mathbf{y}^{\boldsymbol{\alpha}} = \prod_{i=1}^{\infty} y_i^{\alpha_i}. \quad (2.16b)$$

Obviously, (2.16a) is valid for $s = 1$. Taking derivative for (2.16a) w.r.t. y_k and supposing (2.16a) is right for $s - 1$, one finds

$$\frac{\partial}{\partial y_k} l.h.s. \text{ of (2.16a)} = s \left(\sum_{j=1}^{\infty} y_j \right)^{s-1} = s \cdot (s-1)! \sum_{|\alpha|=s-1} \frac{y^\alpha}{\alpha!} = s! \sum_{|\alpha|=s-1} \frac{y^\alpha}{\alpha!},$$

and meanwhile

$$\frac{\partial}{\partial y_k} r.h.s. \text{ of (2.16a)} = s! \sum_{|\alpha|=s} \frac{y_k^{\alpha_k-1}}{(\alpha_k-1)!} \prod_{j \neq k} \frac{y_j^{\alpha_j}}{\alpha_j!} = s! \sum_{|\alpha|=s-1} \frac{y^\alpha}{\alpha!}.$$

That means (2.16a) is valid for any $s \in \mathbb{Z}^+$. Then, noting that

$$\ln \left(1 + \sum_{i=1}^{\infty} t_i k^i \right) = \sum_{s=1}^{\infty} (-1)^{s-1} \frac{1}{s} \left(\sum_{i=1}^{\infty} t_i k^i \right)^s,$$

using (2.16a) with $y_i = t_i k^i$, and rearranging the expansion in terms of k , we get (2.13a). \square

With this Proposition, from the formal conservation law (2.7) we have

Proposition 3. *When θ and η are defined by (2.10) and (2.11), respectively, the formal conservation law (2.7) yields infinitely many conservation laws,*

$$\Delta_m \ln \rho = \Delta_n \ln \omega, \quad (2.17a)$$

$$\Delta_m h_s(\theta) = \Delta_n h_s(\rho \sigma \underline{\theta}), \quad (s = 1, 2, 3, \dots), \quad (2.17b)$$

where $h_s(\mathbf{t})$ is defined in (2.13b),

$$\theta = (\theta_1, \theta_2, \dots), \quad \underline{\theta} = (1, \theta_1, \theta_2, \dots), \quad (2.17c)$$

and $\rho \sigma \underline{\theta} = (\sigma \rho, \sigma \rho \theta_1, \sigma \rho \theta_2, \dots)$.

3 Conservation laws of ABS lattice equations

3.1 Conservation laws for H1, H2, H3, Q1, Q2, Q3 and A1 equation

Let us first, taking H1 equation as an example, give a detailed procedure of deriving infinitely many conservation laws.

The Lax pair of H1 equation reads

$$\tilde{\phi} = \frac{\beta}{\sqrt{r-p}} \begin{pmatrix} u & -u\tilde{u} + p - r \\ 1 & -\tilde{u} \end{pmatrix} \phi, \quad (3.1a)$$

$$\hat{\phi} = \frac{\gamma}{\sqrt{r-q}} \begin{pmatrix} u & -u\hat{u} + q - r \\ 1 & -\hat{u} \end{pmatrix} \phi. \quad (3.1b)$$

Taking $\beta = \sqrt{r-p} = \varepsilon$, from (3.1a) we can find

$$\tilde{\tilde{\phi}}_2 = (u - \tilde{\tilde{u}}) \tilde{\phi}_2 - \varepsilon^2 \phi_2, \quad (3.2)$$

which leads to

$$\tilde{\theta}\theta = (u - \tilde{u})\theta - \varepsilon^2, \quad (3.3)$$

where θ is defined in (2.5), i.e., $\theta = \tilde{\phi}_2/\phi_2$. This is the discrete Riccati equation (2.9) with $\mu = 1$ and is solved by (2.10). Further, taking $\gamma = \sqrt{r - q}$, from the Lax pair (3.1) we have

$$\theta = \frac{\tilde{\phi}_2}{\phi_2} = \frac{\phi_1}{\phi_2} - \tilde{u}, \quad \eta = \frac{\hat{\phi}_2}{\phi_2} = \frac{\phi_1}{\phi_2} - \hat{u}, \quad (3.4)$$

which by eliminating ϕ_1/ϕ_2 yields the relation

$$\eta = \theta + \tilde{u} - \hat{u}. \quad (3.5)$$

This is (2.11) with $\omega = \tilde{u} - \hat{u}$ and $\sigma = 1/(\tilde{u} - \hat{u})$. Thus, for H1 equation, based on Proposition 3 we can write out infinitely many conservation laws (2.17) with μ , ω , σ obtained above. We note that these conservation laws are as same as those derived via Gardner method [15]

For the lattice equations H2, H3, A1, Q1, Q2, and Q3, starting from their Lax pairs, we can also derive infinitely many conservation laws through a similar procedure. Let us skip details and list main results of these equations together with H1 equation.

Proposition 4. *For the lattice equations H1, H2, H3, A1, Q1, Q2, and Q3 in ABS list, starting from their Lax pairs, one can construct a formal conservation law (2.7) with θ and η defined in (2.5), where θ satisfies the discrete Riccati equation (2.9) solved by (2.10) and η is expressed through (2.11). By means of the polynomials $\{h_j(\mathbf{t})\}$ defined in (2.13b), one can explicitly express the infinitely many conservation laws as (2.17). In the following for H1, H2, H3, A1, Q1, Q2, and Q3 equation, we list out their Lax pairs, parametrisation of β , γ , and auxiliary functions μ , ω and σ . For H1 equation, its Lax pair reads (3.1) with*

$$\beta = \sqrt{r - p} = \varepsilon, \quad \gamma = \sqrt{r - q}, \quad \mu = 1, \quad \omega = \tilde{u} - \hat{u}, \quad \sigma = \frac{1}{\tilde{u} - \hat{u}}. \quad (3.6)$$

For H2 equation, its Lax pair reads

$$\tilde{\phi} = \frac{\beta}{\sqrt{2(r-p)(p+u+\tilde{u})}} \begin{pmatrix} u+p-r & -u\tilde{u} + (p-r)(u+\tilde{u}) + p^2 - r^2 \\ 1 & -\tilde{u} - p + r \end{pmatrix} \phi, \quad (3.7a)$$

$$\hat{\phi} = \frac{\gamma}{\sqrt{2(r-q)(q+u+\hat{u})}} \begin{pmatrix} u+q-r & -u\hat{u} + (q-r)(u+\hat{u}) + q^2 - r^2 \\ 1 & -\hat{u} - q + r \end{pmatrix} \phi, \quad (3.7b)$$

and

$$\beta = \sqrt{2(r-p)} = \varepsilon, \quad \gamma = \sqrt{2(r-q)}, \quad (3.8a)$$

$$\mu = \sqrt{p+u+\tilde{u}}, \quad \omega = \frac{p-q+\tilde{u}-\hat{u}}{\sqrt{q+u+\hat{u}}}, \quad \sigma = \frac{\sqrt{p+u+\tilde{u}}}{p-q+\tilde{u}-\hat{u}}. \quad (3.8b)$$

For H3 equation, its Lax pair reads

$$\tilde{\phi} = \frac{\beta}{\sqrt{(p^2-r^2)(u\tilde{u}+p\delta)}} \begin{pmatrix} ru & -pu\tilde{u} - \delta(p^2-r^2) \\ p & -r\tilde{u} \end{pmatrix} \phi, \quad (3.9a)$$

$$\hat{\phi} = \frac{\gamma}{\sqrt{(q^2-r^2)(u\hat{u}+q\delta)}} \begin{pmatrix} ru & -qu\hat{u} - \delta(q^2-r^2) \\ q & -r\hat{u} \end{pmatrix} \phi, \quad (3.9b)$$

and

$$\beta = \frac{\sqrt{p^2 - r^2}}{r} = \varepsilon, \quad \gamma = \frac{\sqrt{q^2 - r^2}}{r}, \quad (3.10a)$$

$$\mu = \sqrt{u\tilde{u} + p\delta}, \quad \omega = \frac{q\tilde{u} - p\hat{u}}{p\sqrt{u\hat{u} + q\delta}}, \quad \sigma = \frac{q\sqrt{u\tilde{u} + p\delta}}{q\tilde{u} - p\hat{u}}. \quad (3.10b)$$

For Q1 equation, its Lax pair reads

$$\tilde{\phi} = \frac{\beta}{\sqrt{r(r-p)[(u-\tilde{u})^2 - \delta^2 p^2]}} \begin{pmatrix} ru + (p-r)\tilde{u} & -pu\tilde{u} - \delta^2 pr(p-r) \\ p & (r-p)u - r\tilde{u} \end{pmatrix} \phi, \quad (3.11a)$$

$$\hat{\phi} = \frac{\gamma}{\sqrt{r(r-q)[(u-\hat{u})^2 - \delta^2 q^2]}} \begin{pmatrix} ru + (q-r)\hat{u} & -qu\hat{u} - \delta^2 qr(q-r) \\ q & (r-q)u - r\hat{u} \end{pmatrix} \phi, \quad (3.11b)$$

and

$$\beta = \frac{\sqrt{r(r-p)}}{r} = \varepsilon, \quad \gamma = \frac{\sqrt{r(r-q)}}{r}, \quad (3.12a)$$

$$\mu = \sqrt{(u-\tilde{u})^2 - \delta^2 p^2}, \quad \omega = \frac{q(\tilde{u}-u) - p(\hat{u}-u)}{p\sqrt{(u-\hat{u})^2 - \delta^2 q^2}}, \quad \sigma = \frac{q\sqrt{(u-\tilde{u})^2 - \delta^2 p^2}}{q(\tilde{u}-u) - p(\hat{u}-u)}. \quad (3.12b)$$

For Q2 equation, its Lax pair reads

$$\tilde{\phi} = \frac{\beta}{\sqrt{A}} \begin{pmatrix} ru + (p-r)\tilde{u} - pr(p-r) & -pu\tilde{u} - pr(p-r)(u+\tilde{u}-p^2+pr-r^2) \\ p & (r-p)u - r\tilde{u} + pr(p-r) \end{pmatrix} \phi, \quad (3.13a)$$

$$\hat{\phi} = \frac{\gamma}{\sqrt{B}} \begin{pmatrix} ru + (q-r)\hat{u} - qr(q-r) & -qu\hat{u} - qr(q-r)(u+\hat{u}-q^2+qr-r^2) \\ q & (r-q)u - r\hat{u} + qr(q-r) \end{pmatrix} \phi, \quad (3.13b)$$

with

$$A = r(r-p)[(p^2-u)^2 + \tilde{u}(\tilde{u}-2u-2p^2)], \quad B = r(r-q)[(q^2-u)^2 + \hat{u}(\hat{u}-2u-2q^2)], \quad (3.13c)$$

and we take

$$\beta = \frac{\sqrt{r(r-p)}}{r} = \varepsilon, \quad \gamma = \frac{\sqrt{r(r-q)}}{r}, \quad \mu = \sqrt{(p^2-u)^2 + \tilde{u}(\tilde{u}-2u-2p^2)}, \quad (3.14a)$$

$$\omega = \frac{(p-q)(u-pq) + q\tilde{u} - p\hat{u}}{p\sqrt{(q^2-u)^2 + \hat{u}(\hat{u}-2u-2q^2)}}, \quad \sigma = \frac{q\sqrt{(p^2-u)^2 + \tilde{u}(\tilde{u}-2u-2p^2)}}{(p-q)(u-pq) + q\tilde{u} - p\hat{u}}. \quad (3.14b)$$

For Q3 equation, its Lax pair reads

$$\tilde{\phi} = \frac{\beta}{\sqrt{A}} \begin{pmatrix} p(r^2-1)u - (r^2-p^2)\tilde{u} & -r(p^2-1)u\tilde{u} + \delta^2(p^2-r^2)(p^2-1)(r^2-1)/4pr \\ r(p^2-1) & (r^2-p^2)u - p(r^2-1)\tilde{u} \end{pmatrix} \phi, \quad (3.15a)$$

$$\hat{\phi} = \frac{\gamma}{\sqrt{B}} \begin{pmatrix} q(r^2-1)u - (r^2-q^2)\hat{u} & -r(q^2-1)u\hat{u} + \delta^2(q^2-r^2)(q^2-1)(r^2-1)/4qr \\ r(q^2-1) & (r^2-q^2)u - q(r^2-1)\hat{u} \end{pmatrix} \phi, \quad (3.15b)$$

with

$$A = (r^2 - 1)(r^2 - p^2)[(u - p\tilde{u})(pu - \tilde{u}) + \delta^2(1 - p^2)^2/(4p)], \quad (3.15c)$$

$$B = (r^2 - 1)(r^2 - q^2)[(u - q\hat{u})(qu - \hat{u}) + \delta^2(1 - q^2)^2/(4q)], \quad (3.15d)$$

and we take

$$\beta = \frac{\sqrt{(r^2 - 1)(r^2 - p^2)}}{r^2 - 1} = \varepsilon, \quad \gamma = \frac{\sqrt{(r^2 - 1)(r^2 - q^2)}}{r^2 - 1}, \quad (3.16a)$$

$$\mu = \sqrt{(u - p\tilde{u})(pu - \tilde{u}) + \delta^2(1 - p^2)^2/(4p)}, \quad (3.16b)$$

$$\omega = \frac{p(q^2 - 1)\tilde{u} + (p^2 - q^2)u - q(p^2 - 1)\hat{u}}{(p^2 - 1)\sqrt{(u - q\hat{u})(qu - \hat{u}) + \delta^2(1 - q^2)^2/(4q)}}, \quad (3.16c)$$

$$\sigma = \frac{(q^2 - 1)\sqrt{(u - p\tilde{u})(pu - \tilde{u}) + \delta^2(1 - p^2)^2/(4p)}}{p(q^2 - 1)\tilde{u} + (p^2 - q^2)u - q(p^2 - 1)\hat{u}}. \quad (3.16d)$$

For A1 equation, its Lax pair reads

$$\tilde{\phi} = \frac{\beta}{\sqrt{r(p - r)[(u + \tilde{u})^2 - \delta^2 p^2]}} \begin{pmatrix} ru + (r - p)\tilde{u} & -pu\tilde{u} + \delta^2 pr(p - r) \\ p & (p - r)u - r\tilde{u} \end{pmatrix} \phi, \quad (3.17a)$$

$$\hat{\phi} = \frac{\gamma}{\sqrt{r(q - r)[(u + \hat{u})^2 - \delta^2 q^2]}} \begin{pmatrix} ru + (r - q)\hat{u} & -qu\hat{u} + \delta^2 qr(q - r) \\ q & (q - r)u - r\hat{u} \end{pmatrix} \phi, \quad (3.17b)$$

and

$$\beta = \frac{\sqrt{r(p - r)}}{r} = \varepsilon, \quad \gamma = \frac{\sqrt{r(q - r)}}{r}, \quad (3.18a)$$

$$\mu = \sqrt{(u + \tilde{u})^2 - \delta^2 p^2}, \quad \omega = \frac{q(u + \tilde{u}) - p(u + \hat{u})}{p\sqrt{(u + \hat{u})^2 - \delta^2 q^2}}, \quad \sigma = \frac{q\sqrt{(u + \tilde{u})^2 - \delta^2 p^2}}{q(u + \tilde{u}) - p(u + \hat{u})}. \quad (3.18b)$$

There is a transformation [1]

$$u = (-1)^{n+m}v \quad (3.19)$$

connecting Q1 equation and A1 equation

$$p(v + \tilde{v})(\tilde{v} + \hat{\tilde{v}}) - q(v + \tilde{v})(\hat{v} + \hat{\tilde{v}}) - \delta^2 pq(p - q) = 0. \quad (3.20)$$

In fact, one can substitute (3.19) into the infinitely many conservation laws of Q1 equation to get those of A1 equation. The obtained conservation laws are as same as those derived through (3.18) (with v in place of u).

3.2 Conservation laws for A2 equation

3.2.1 Transformation

The transformation [1]

$$u = v(-1)^{n+m} \quad (3.21)$$

connects $Q3|_{\delta=0}$ equation

$$(q^2 - p^2)(u\hat{u} + \tilde{u}\hat{u}) + q(p^2 - 1)(u\tilde{u} + \hat{u}\hat{u}) - p(q^2 - 1)(u\hat{u} + \tilde{u}\hat{u}) = 0 \quad (3.22)$$

and A2 equation

$$(q^2 - p^2)(v\hat{v}\hat{v} + 1) + q(p^2 - 1)(v\hat{v} + \hat{v}\hat{v}) - p(q^2 - 1)(v\hat{v} + \hat{v}\hat{v}) = 0. \quad (3.23)$$

Noting that the conservation law (2.4) of equation (2.1) is a relation that holds for all of u satisfying (2.1), for A2 equation (3.23) what we need is to list out conservation laws of $Q3|_{\delta=0}$ equation (3.22) and then replace u by $v^{(-1)^{n+m}}$.

Proposition 5. *The infinitely many conservation laws of $Q3|_{\delta=0}$ equation (3.22) is given by (2.17) with θ , η and $\{h_j(\mathbf{t})\}$ given in (2.10), (2.11) and (2.13b), respectively, and*

$$\mu = \sqrt{(u - p\tilde{u})(pu - \tilde{u})}, \quad (3.24a)$$

$$\omega = \frac{p(q^2 - 1)\tilde{u} + (p^2 - q^2)u - q(p^2 - 1)\hat{u}}{(p^2 - 1)\sqrt{(u - q\hat{u})(qu - \hat{u})}}, \quad \sigma = \frac{(q^2 - 1)\sqrt{(u - p\tilde{u})(pu - \tilde{u})}}{p(q^2 - 1)\tilde{u} + (p^2 - q^2)u - q(p^2 - 1)\hat{u}}. \quad (3.24b)$$

The infinitely many conservation laws of A2 equation (3.23) can be given through the infinitely many conservation laws of $Q3|_{\delta=0}$ equation (3.22) by replacing u by $v^{(-1)^{n+m}}$.

3.2.2 Lax pair approach

Conservation laws of A2 equation can also be derived directly from its Lax pair which reads

$$\tilde{\phi} = \frac{\beta}{\sqrt{A}} \begin{pmatrix} -r(p^2 - 1)u & p(r^2 - 1)u\tilde{u} - (r^2 - p^2) \\ (r^2 - p^2)u\tilde{u} - p(r^2 - 1) & r(p^2 - 1)\tilde{u} \end{pmatrix} \phi, \quad (3.25a)$$

$$\hat{\phi} = \frac{\gamma}{\sqrt{B}} \begin{pmatrix} -r(q^2 - 1)u & q(r^2 - 1)u\hat{u} - (r^2 - q^2) \\ (r^2 - q^2)u\hat{u} - q(r^2 - 1) & r(q^2 - 1)\hat{u} \end{pmatrix} \phi, \quad (3.25b)$$

with

$$A = (r^2 - 1)(r^2 - p^2)(p - u\tilde{u})(pu\tilde{u} - 1), \quad B = (r^2 - 1)(r^2 - q^2)(q - u\hat{u})(qu\hat{u} - 1).$$

Since, in this case, the equation (2.8) becomes

$$\sqrt{\tilde{A}}\tilde{\phi}_2 = \beta \frac{rp(p^2 - 1)(r^2 - 1)(u - \tilde{u})}{p(1 - r^2) + (r^2 - p^2)u\tilde{u}} \tilde{\phi}_2 - \beta^2 \sqrt{A} \frac{p(1 - r^2) + (r^2 - p^2)u\tilde{u}}{p(1 - r^2) + (r^2 - p^2)u\tilde{u}} \phi_2, \quad (3.26)$$

which is difficult to get a solvable Riccati equation for $\theta = \tilde{\phi}_2/\phi_2$, we turn to another formulae set

$$\theta = \frac{\tilde{\phi}_2}{\phi_2} = \frac{1}{\mu} [[(r^2 - p^2)u\tilde{u} - p(r^2 - 1)]\zeta + r(p^2 - 1)\tilde{u}], \quad (3.27a)$$

$$\eta = \frac{\hat{\phi}_2}{\phi_2} = \frac{1}{\nu} [[(r^2 - q^2)u\hat{u} - q(r^2 - 1)]\zeta + r(q^2 - 1)\hat{u}]. \quad (3.27b)$$

By θ and η the formal conservation law is written as

$$\Delta_m \ln \theta = \Delta_n \ln \eta.$$

(3.27a) and (3.27b) are derived from the Lax pair (3.25), where

$$\zeta = \frac{\phi_1}{\phi_2}, \quad (3.27c)$$

and

$$\mu = \sqrt{(p - u\tilde{u})(pu\tilde{u} - 1)}, \quad \nu = \sqrt{(q - u\hat{u})(qu\hat{u} - 1)}, \quad (3.27d)$$

and we have taken

$$\beta = \sqrt{(r^2 - 1)(r^2 - p^2)}, \quad \gamma = \sqrt{(r^2 - 1)(r^2 - q^2)}.$$

ζ is determined by the following equation

$$\tilde{\zeta} = \frac{-r(p^2 - 1)u\zeta + p(r^2 - 1)u\tilde{u} - (r^2 - p^2)}{[(r^2 - p^2)u\tilde{u} - p(r^2 - 1)]\zeta + r(p^2 - 1)\tilde{u}}, \quad (3.28)$$

which is derived from (3.25a).

To solve (3.28), we take (cf. [16])

$$\zeta = \tilde{u} + \xi, \quad \varepsilon = r - p, \quad (3.29)$$

and we reach to

$$(a_0 + a_1\varepsilon + a_2\varepsilon^2)\xi\tilde{\xi} + (b_1\varepsilon + b_2\varepsilon^2)\tilde{\xi} + (c_0 + c_1\varepsilon + c_2\varepsilon^2)\xi + (d_1\varepsilon + d_2\varepsilon^2) = 0, \quad (3.30)$$

where

$$\begin{aligned} a_0 &= -p(p^2 - 1), \quad a_1 = 2p(u\tilde{u} - p), \quad a_2 = u\tilde{u} - p, \\ b_1 &= (2pu\tilde{u} - p^2 - 1)\tilde{u}, \quad b_2 = (u\tilde{u} - p)\tilde{u}, \\ c_0 &= p(p^2 - 1)(u - \tilde{u}), \quad c_1 = 2p\tilde{u}(u\tilde{u} - p) + (p^2 - 1)u, \quad c_2 = (u\tilde{u} - p)\tilde{u}, \\ d_1 &= 2p(u\tilde{u}^2\tilde{u} + 1) - (p^2 + 1)(u\tilde{u} + \tilde{u}\tilde{u}), \quad d_2 = u\tilde{u}^2\tilde{u} + 1 - p(u\tilde{u} + \tilde{u}\tilde{u}). \end{aligned}$$

Equation (3.30) is then solved by

$$\xi = \sum_{j=1}^{\infty} \xi_j \varepsilon^j,$$

with

$$\begin{aligned} \xi_1 &= -\frac{d_1}{c_0}, \\ \xi_2 &= -\frac{1}{c_0}(a_0\xi_1\tilde{\xi}_1 + b_1\tilde{\xi}_1 + c_1\xi_1 + d_2), \\ \xi_3 &= -\frac{1}{c_0}[a_0(\xi_1\tilde{\xi}_2 + \xi_2\tilde{\xi}_1) + a_1\xi_1\tilde{\xi}_1 + b_1\tilde{\xi}_2 + b_2\tilde{\xi}_1 + c_1\xi_2 + c_2\xi_1], \\ \xi_s &= -\frac{1}{c_0}\left(\sum_{k=0}^2 a_k \sum_{i=1}^{s-k-1} \xi_i \tilde{\xi}_{s-k-i} + \sum_{k=1}^2 b_k \tilde{\xi}_{s-k} + \sum_{k=1}^2 c_k \xi_{s-k}\right), \quad (s = 4, 5, \dots). \end{aligned}$$

Next, we express (3.27) in terms of ξ and ε as

$$\theta = \frac{1}{\mu} [(f_0 + f_1\varepsilon + f_2\varepsilon^2)\xi + g_1\varepsilon + g_2\varepsilon^2], \quad (3.33a)$$

$$\eta = \frac{1}{\nu} [(w_0 + w_1\varepsilon + w_2\varepsilon^2)\xi + z_0 + z_1\varepsilon + z_2\varepsilon^2], \quad (3.33b)$$

with

$$\begin{aligned} f_0 &= p(1 - p^2), \quad f_1 = 2p(u\tilde{u} - p), \quad f_2 = u\tilde{u} - p, \\ g_1 &= (2pu\tilde{u} - p^2 - 1)\tilde{u}, \quad g_2 = (u\tilde{u} - p)\tilde{u}, \\ w_0 &= (p^2 - q^2)u\hat{u} - q(p^2 - 1), \quad w_1 = 2p(u\hat{u} - q), \quad w_2 = u\hat{u} - q, \\ z_0 &= (p^2 - q^2)u\hat{u}\tilde{u} - q(p^2 - 1)\tilde{u} + p(q^2 - 1)\hat{u}, \quad z_1 = 2p\tilde{u}(u\hat{u} - q) + (q^2 - 1)\hat{u}, \quad z_2 = (u\hat{u} - q)\tilde{u}, \end{aligned}$$

and we can get solutions

$$\theta = \frac{(f_0\xi_1 + g_1)\varepsilon}{\mu} \left(1 + \sum_{j=1}^{\infty} \theta_j \varepsilon^j \right), \quad (3.34)$$

with

$$\theta_1 = \frac{f_0\xi_2 + f_1\xi_1 + g_2}{f_0\xi_1 + g_1}, \quad (3.35a)$$

$$\theta_s = \frac{f_0\xi_{s+1} + f_1\xi_s + f_2\xi_{s-1}}{f_0\xi_1 + g_1}, \quad (s = 2, 3, \dots), \quad (3.35b)$$

and

$$\eta = \frac{z_0}{\nu} \left(1 + \sum_{j=1}^{\infty} \eta_j \varepsilon^j \right), \quad (3.36)$$

with

$$\eta_1 = \frac{1}{z_0} (w_0\xi_1 + z_1), \quad (3.37a)$$

$$\eta_2 = \frac{1}{z_0} (w_0\xi_2 + w_1\xi_1 + z_2), \quad (3.37b)$$

$$\eta_s = \frac{1}{z_0} \sum_{i=0}^2 w_i \xi_{s-i}, \quad (s = 3, 4, \dots). \quad (3.37c)$$

Finally, by means of the polynomials $\{h_j(\mathbf{t})\}$ defined in (2.13b), the infinitely many conservation laws of A2 equation are given by

$$\Delta_m \ln \frac{f_0\xi_1 + g_1}{\mu} = \Delta_n \ln \frac{z_0}{\nu}, \quad (3.38a)$$

$$\Delta_m h_s(\boldsymbol{\theta}) = \Delta_n h_s(\boldsymbol{\eta}), \quad s = 1, 2, \dots, \quad (3.38b)$$

where

$$\boldsymbol{\theta} = (\theta_1, \theta_2, \dots), \quad \boldsymbol{\eta} = (\eta_1, \eta_2, \dots),$$

with μ , ν , $\{\theta_j\}$ and $\{\eta_j\}$ given in (3.27d), (3.35) and (3.37) respectively.

3.3 Conservation laws for Q4 equation

For Q4 equation, one can use the same method as in Sec.3.2.2. Lax pair of Q4 equation is

$$\tilde{\phi} = \frac{\beta}{\sqrt{A}} \begin{pmatrix} r(1-p^2r^2)u + (pR-rP)\tilde{u} & -p(1-p^2r^2)u\tilde{u} - pr(pR-rP) \\ p(1-p^2r^2) + pr(pR-rP)u\tilde{u} & -r(1-p^2r^2)\tilde{u} - (pR-rP)u \end{pmatrix} \phi, \quad (3.39a)$$

$$\hat{\phi} = \frac{\gamma}{\sqrt{B}} \begin{pmatrix} r(1-q^2r^2)u + (qR-rQ)\hat{u} & -q(1-q^2r^2)u\hat{u} - qr(qR-rQ) \\ q(1-q^2r^2) + qr(qR-rQ)u\hat{u} & -r(1-q^2r^2)\hat{u} - (qR-rQ)u \end{pmatrix} \phi, \quad (3.39b)$$

with

$$\begin{aligned} A &= r(1-p^2r^2)(pR-rP)[2Pu\tilde{u} + p^2(u^2\tilde{u}^2 + 1) - \tilde{u}^2 - u^2], \\ B &= r(1-q^2r^2)(qR-rQ)[2Qu\hat{u} + q^2(u^2\hat{u}^2 + 1) - \hat{u}^2 - u^2], \end{aligned}$$

and (r, R) are formulated by the elliptic curve

$$R^2 = r^4 - kr^2 + 1. \quad (3.39c)$$

Taking $\beta = \sqrt{r(1-p^2r^2)(pR-rP)}$ and $\gamma = \sqrt{r(1-q^2r^2)(qR-rQ)}$ in (3.39), we have

$$\theta = \frac{1}{\mu} [[p(1-p^2r^2) + pr(pR-rP)u\tilde{u}]\zeta - r(1-p^2r^2)\tilde{u} - (pR-rP)u], \quad (3.40a)$$

$$\eta = \frac{1}{\nu} [[q(1-q^2r^2) + qr(qR-rQ)u\hat{u}]\zeta - r(1-q^2r^2)\hat{u} - (qR-rQ)u], \quad (3.40b)$$

where

$$\zeta = \frac{\phi_1}{\phi_2}, \quad (3.40c)$$

and

$$\mu = \sqrt{2Pu\tilde{u} + p^2(u^2\tilde{u}^2 + 1) - \tilde{u}^2 - u^2}, \quad \nu = \sqrt{2Qu\hat{u} + q^2(u^2\hat{u}^2 + 1) - \hat{u}^2 - u^2}. \quad (3.40d)$$

The formal conservation law is given by

$$\Delta_m \ln \theta = \Delta_n \ln \eta. \quad (3.41)$$

ζ is determined by the following Riccati equation

$$\tilde{\zeta}(c\zeta + d) = a\zeta + b, \quad (3.42)$$

which is derived from (3.39a), and here

$$\begin{aligned} a &= r(1-p^2r^2)u + (pR-rP)\tilde{u}, \quad b = -p(1-p^2r^2)u\tilde{u} - pr(pR-rP), \\ c &= p(1-p^2r^2) + pr(pR-rP)u\tilde{u}, \quad d = -r(1-p^2r^2)\tilde{u} - (pR-rP)u. \end{aligned}$$

To solve it we take (cf. [16])

$$\zeta = \tilde{u} + \xi, \quad \varepsilon = r - p, \quad (3.43)$$

and expand a, b, c, d as

$$a = \sum_{i=0}^{\infty} a_i \varepsilon^i, \quad b = \sum_{i=0}^{\infty} b_i \varepsilon^i, \quad c = \sum_{i=0}^{\infty} c_i \varepsilon^i, \quad d = \sum_{i=0}^{\infty} d_i \varepsilon^i. \quad (3.44)$$

Since R defined in (3.39c) can be expanded as

$$R = \sum_{i=0}^{\infty} r_i \varepsilon^i, \quad (3.45)$$

in which r_i is given by

$$r_i = P \sum_{\|\alpha\|=i} \frac{\mathbf{g}^\alpha}{\alpha!} \prod_{i=0}^{|\alpha|-1} \left(\frac{1}{2} - i\right),$$

where

$$\begin{aligned} \mathbf{g} &= (g_1, g_2, g_3, g_4), \quad \alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4), \quad \alpha_i \in \{0, 1, 2, \dots\} \\ g_1 &= \frac{2}{P}(2p^3 - kp), \quad g_2 = \frac{1}{P}(6p^2 - k), \quad g_3 = \frac{4p}{P}, \quad g_4 = \frac{1}{P}, \\ \|\alpha\| &= \sum_{i=1}^4 j\alpha_j, \quad |\alpha| = \sum_{i=1}^4 \alpha_i, \quad \alpha! = \prod_{i=1}^4 (\alpha_i!), \quad \mathbf{g}^\alpha = \prod_{i=1}^4 g_i^{\alpha_i}, \end{aligned}$$

we have

$$\begin{aligned} a_0 &= p(1 - p^4)u, \quad a_1 = (1 - 3p^4)u + (pr_1 - P)\tilde{u}, \quad a_2 = -3p^3u + pr_2\tilde{u}, \quad a_3 = -p^2u + pr_3\tilde{u}, \\ a_i &= pr_i\tilde{u}, \quad (i = 4, 5, \dots), \\ b_0 &= -p(1 - p^4)u\tilde{u}, \quad b_1 = 2p^4u\tilde{u} - p^2(pr_1 - P), \quad b_2 = p^3u\tilde{u} - p(pr_1 - P) - p^3r_2, \\ b_i &= -p^2(pr_i + r_{i-1}), \quad (i = 3, 4, \dots), \\ c_0 &= p(1 - p^4), \quad c_1 = p^2(pr_1 - P)u\tilde{u} - 2p^4, \quad c_2 = [p(pr_1 - P) + p^3r_2]u\tilde{u} - p^3, \\ c_i &= p^2(pr_i + r_{i-1})u\tilde{u}, \quad (i = 3, 4, \dots), \\ d_0 &= -p(1 - p^4)\tilde{u}, \quad d_1 = -(1 - 3p^4)\tilde{u} - (pr_1 - P)u, \quad d_2 = 3p^3\tilde{u} - pr_2u, \quad d_3 = p^2\tilde{u} - pr_3u, \\ d_i &= -pr_iu, \quad (i = 4, 5, \dots). \end{aligned}$$

Consequently, (3.42) turns out to be

$$\sum_{i=0}^{\infty} c_i \varepsilon^i \xi \tilde{\xi} + \sum_{i=1}^{\infty} (c_i \tilde{u} + d_i) \varepsilon^i \tilde{\xi} + \sum_{i=0}^{\infty} (c_i \tilde{u} - a_i) \varepsilon^i \xi + \sum_{i=1}^{\infty} [(c_i \tilde{u} + d_i) \tilde{u} - a_i \tilde{u} - b_i] \varepsilon^i = 0. \quad (3.46)$$

This is solved by

$$\xi = \sum_{j=1}^{\infty} \xi_j \varepsilon^j,$$

with

$$\begin{aligned} \xi_1 &= \frac{-1}{c_0 \tilde{u} - a_0} [(c_1 \tilde{u} + d_1) \tilde{u} - a_1 \tilde{u} - b_1], \\ \xi_s &= \frac{-1}{c_0 \tilde{u} - a_0} \left[\sum_{k=0}^{s-2} \sum_{i=1}^{s-k-1} c_k \xi_i \tilde{\xi}_{s-k-i} + \sum_{i=1}^{s-1} [(c_i \tilde{u} + d_i) \tilde{\xi}_{s-i} + (c_i \tilde{u} - a_i) \xi_{s-i}] + (c_s \tilde{u} + d_s) \tilde{u} - a_s \tilde{u} - b_s \right], \end{aligned}$$

for $s = 2, 3, \dots$.

Next, from (3.43) and (3.40) one has

$$\theta = \frac{1}{\mu} \left(\sum_{i=0}^{\infty} c_i \varepsilon^i \xi + \sum_{i=0}^{\infty} f_i \varepsilon^i \right), \quad (3.48a)$$

$$\eta = \frac{1}{\nu} \left(\sum_{i=0}^{\infty} w_i \varepsilon^i \xi + \sum_{i=0}^{\infty} z_i \varepsilon^i \right), \quad (3.48b)$$

where

$$\begin{aligned} f_0 &= 0, \quad f_i = c_i \tilde{u} + d_i, \quad (i = 1, 2, \dots), \\ w_0 &= q(1 - q^2 p^2) + qp(qP - pQ)u\hat{u}, \quad w_1 = -2pq^3 + q(qP + pqr_1 - 2pQ)u\hat{u}, \\ w_2 &= -q^3 + q(qpr_2 + qr_1 - Q)u\hat{u}, \quad w_i = q^2(pr_i + r_{i-1})u\hat{u}, \quad (i = 3, 4, \dots), \\ z_0 &= -p(1 - q^2 p^2)\hat{u} - (qP - pQ)u + w_0\tilde{u}, \quad z_1 = (3p^2 q^2 - 1)\hat{u} - (qr_1 - Q)u + w_1\tilde{u}, \\ z_2 &= 3pq^2\hat{u} - qr_2u + w_2\tilde{u}, \quad z_3 = q^2\hat{u} - qr_3u + w_3\tilde{u}, \quad z_i = -qr_iu + w_i\tilde{u}, \quad (i = 4, 5, \dots). \end{aligned}$$

Further

$$\theta = \frac{(c_0 \xi_1 + f_1) \varepsilon}{\mu} \left(1 + \sum_{j=1}^{\infty} \theta_j \varepsilon^j \right), \quad (3.49)$$

where

$$\theta_j = \frac{1}{c_0 \xi_1 + f_1} \left(\sum_{k=0}^j c_k \xi_{j+1-k} + f_{j+1} \right), \quad (j = 1, 2, \dots), \quad (3.50)$$

and

$$\eta = \frac{z_0}{\nu} \left(1 + \sum_{j=1}^{\infty} \eta_j \varepsilon^j \right), \quad (3.51)$$

where

$$\eta_j = \frac{1}{z_0} \left(\sum_{k=0}^{j-1} w_k \xi_{j-k} + z_j \right), \quad (j = 1, 2, \dots). \quad (3.52)$$

Finally, by means of the polynomials $\{h_j(\mathbf{t})\}$ defined in (2.13b), from the formal conservation law (3.41), the infinitely many conservation laws of Q4 equation are given by

$$\Delta_m \ln \frac{c_0 \xi_1 + f_1}{\mu} = \Delta_n \ln \frac{z_0}{\nu}, \quad (3.53a)$$

$$\Delta_m h_s(\boldsymbol{\theta}) = \Delta_n h_s(\boldsymbol{\eta}), \quad s = 1, 2, \dots, \quad (3.53b)$$

where

$$\boldsymbol{\theta} = (\theta_1, \theta_2, \dots), \quad \boldsymbol{\eta} = (\eta_1, \eta_2, \dots),$$

with μ , ν , θ_j and η_j are given by (3.40d), (3.50) and (3.52), respectively.

4 Conclusions

We have shown that infinitely many conservation laws of ABS lattice equations can be derived from their Lax pairs. We generalized the approach used in [9]. From a discrete (two by two) Lax pair it is easy to write out a formal conservation law. We found a generic discrete Riccati equation (2.9) that is shared by H1, H2, H3, Q1, Q2, Q3 and A1 equation. This generic Riccati equation is derived from their Lax pairs. It provides a series-form solution for θ , and with the help of polynomials $\{h_j(\mathbf{t})\}$ defined in (2.13b), the infinitely many conservations laws can be expressed both algebraically and explicitly. We also want to emphasise that the value of β that we choose is important for getting the solvable generic discrete Riccati equation, while in Gardner method β is cancelled in the ratio form $\bar{u} = \phi_1/\phi_2$. Besides, we also note that if we conduct the same procedure starting from (q, \wedge) part of Lax pairs, we will get same conservation laws due to the symmetric property (2.12). A2 and Q4 equation seem to be special and so far we do not know whether their Riccati equations fall in the same generic form (2.9). For them we derive their conservation laws by using the approach used for the Ablowitz-Ladik system [9]. This is closely related to Gardner method because for a CAC equation, its Lax pair is obtained by just taking $\bar{u} = \phi_1/\phi_2$ in its BT. However, starting from Lax pairs gives naturally the formal (initial) conservation law. Compared with [16], our formal conservation laws and the initial conservation laws in [16] are same for H1, H2, H3, Q2 and Q3 equation, while for A1, A2, Q1 and Q4 equation, they are different. Our approach can apply to other multidimensionally consistent systems such as NQC equation [20], discrete Boussinesq type equations [21, 22] and so on.

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